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A new relaxed proximal point procedure and applications to nonlinear variational inclusions

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ABSTRACT

A new relaxed algorithmic procedure based on the notion of A-maximal relaxed monotonicity is developed, and then the convergence analysis in the context of solving a general class of nonlinear inclusion problems is examined. Furthermore, some results involving A-maximal relaxed monotone mappings in a Hilbert space setting are included.

1. Introduction

Let X be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. We intend to consider a general class of nonlinear inclusion problems of the form: find a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.

Among the recent advances on algorithmic developments, Eckstein and Bertsekas [1] relaxed the proximal point algorithm, based on the work of Rockafellar [2], and ended up achieving a weak convergence of the sequence to a unique solution of (1). Eckstein and Bertsekas [1] further generalized the alternating direction method of multipliers for convex programming, while they applied the results obtained to the Douglas–Rachford splitting method for finding the zero of the sum of two monotone operators. Recently, Agarwal and Verma [3] studied the over-relaxed proximal point algorithm for the context of solving some inclusion problems. We are primarily concerned with introducing a new relaxed algorithmic procedure based on the notion of A-maximal relaxed monotonicity (also referred to as A-monotonicity [4] in the literature) for solving general inclusion problems in Hilbert space settings, that is, based on algorithmic advances [1–26]. The concept of A-maximal monotonicity was introduced and studied by the author [26], while examining solutions of variational inclusion problems of the form (1) using the resolvent operator technique. The generalized resolvent operator methods are also applicable to several other fields, including equilibria problems in economics, optimization and control theory, operations research, mathematical finance, management and decision sciences, and mathematical programming. For more detailed literature, we recommend to the reader [1–29].

2. General maximal relaxed monotonicity

In this section we discuss some results based on the basic properties of A-maximal monotonicity (also referred to as A-monotonicity in the literature). Let $M : X \to 2^X$ be a multivalued mapping on X. We shall denote both the map M and its

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graph by M, that is, the set $\{(x, y) : y \in M(x)\}$. This is equivalent to stating that a mapping is any subset M of $X \times X$, and $M(x) = \{y : (x, y) \in M\}$. If M is single-valued, we shall still use $M(x)$ to represent the unique y such that $(x, y) \in M$ rather than the singleton set $\{y\}$. This interpretation will depend much on the context. The domain of a map M is defined (as its projection onto the first argument) by

$$\mathrm{dom}(M) = \{x \in X : \exists y \in X : (x, y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.$$

$\mathrm{dom}(M) = X$ will denote the full domain of M, and the range of M is defined by

$$R(M) = \{y \in X : \exists x \in X : (x, y) \in M\}.$$

The inverse M^{-1} of M is $\{(y, x) : (x, y) \in M\}$. For a real number ρ and a mapping M, let $\rho M = \{(x, \rho y) : (x, y) \in M\}$. If L and M are any mappings, we define

$$L + M = \{(x, y + z) : (x, y) \in L, (x, z) \in M\}.$$

Definition 2.1. Let $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be:

(i) Monotone if

$$\langle u^* - v^*, u - v\rangle \geq 0 \quad \forall (u, u^*), (v, v^*) \in M.$$

(ii) (r)-strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*, u - v\rangle \geq r\|u - v\|^2 \quad \forall (u, u^*), (v, v^*) \in M.$$

(iii) (m)-relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, u - v\rangle \geq (-m)\|u - v\|^2 \quad \forall (u, u^*), (v, v^*) \in M.$$

(iv) Cocoercive if

$$\langle u^* - v^*, u - v\rangle \geq \|u^* - v^*\|^2 \quad \forall (u, u^*), (v, v^*) \in M.$$

(v) (c)-cocoercive if there exists a positive constant c such that

$$\langle u^* - v^*, u - v\rangle \geq c\|u^* - v^*\|^2 \quad \forall (u, u^*), (v, v^*) \in M.$$

Definition 2.2 ([4]). Let $A : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be A-maximal (m)-relaxed monotone if:

(i) M is (m)-relaxed monotone for $m > 0$,
(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Example 2.1. Let $A : X \to X$ be an (r)-strongly monotone mapping on X. Let $f : X \to R$ be a locally Lipschitz functional such that ∂f is (m)-relaxed monotone. Then $A + \partial f$ is $(r - m)$-strongly monotone for $r - m > 0$. Then it follows that $A + \partial f$ is pseudomonotone, which is, in fact, maximal monotone. This is equivalent to stating that ∂f is A-maximal (m)-relaxed monotone.

Definition 2.3 ([4]). Let $A : X \to X$ be an (r)-strongly monotone mapping and let $M : X \to 2^X$ be an A-maximal (m)-relaxed monotone mapping. Then the generalized resolvent operator $J^M_{\rho,A} : X \to X$ is defined by

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Definition 2.4 ([6]). Let $H : X \to X$ be (r)-strongly monotone. The map $M : X \to 2^X$ is said to be H-maximal monotone if:

(i) M is monotone,
(ii) $R(H + \rho M) = X$ for $\rho > 0$.

Definition 2.5. Let $H : X \to X$ be an (r)-strongly monotone mapping and let $M : X \to 2^X$ be an H-monotone mapping. Then the generalized resolvent operator $J^M_{\rho,H} : X \to X$ is defined by

$$J^M_{\rho,H}(u) = (H + \rho M)^{-1}(u).$$

Proposition 2.1 ([4]). *Let $A : X \to X$ be a (r)-strongly monotone single-valued mapping and let $M : X \to 2^X$ be an A-maximal (m)-relaxed monotone mapping. Then $(A + \rho M)$ is maximal monotone for $0 < \rho < \frac{r}{m}$.*

Proposition 2.2 ([4]). *Let $A : X \to X$ be an (r)-strongly monotone mapping and let $M : X \to 2^X$ be an A-maximal (m)-relaxed monotone mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued for $0 < \rho < \frac{r}{m}$.*

Proposition 2.3 ([6]). *Let $H : X \to X$ be a (r)-strongly monotone single-valued mapping and let $M : X \to 2^X$ be an H-maximal monotone mapping. Then $(H + \rho M)$ is maximal monotone for $\rho > 0$.*

Proposition 2.4. *Let $H : X \to X$ be an (r)-strongly monotone mapping and let $M : X \to 2^X$ be an H-maximal monotone mapping. Then the operator $(H + \rho M)^{-1}$ is single-valued for $\rho > 0$.*

Proposition 2.5 ([4]). *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be A-maximal (m)-relaxed monotone. Then the generalized resolvent operator associated with M and defined by*

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u) \quad \forall u \in X,$$

is $(\frac{1}{r-\rho m})$-Lipschitz continuous, where $r - \rho m > 0$.

Proposition 2.6 ([6]). *Let X be a real Hilbert space, let $H : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be H-maximal monotone. Then the generalized resolvent operator associated with M and defined by*

$$J^M_{\rho,H}(u) = (H + \rho M)^{-1}(u) \quad \forall u \in X,$$

is $(\frac{1}{r})$-Lipschitz continuous for $\rho > 0$.

Proposition 2.7. *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone (s)-Lipschitz continuous, and let $M : X \to 2^X$ be A-maximal (m)-relaxed monotone. Then the generalized resolvent operator associated with M satisfies*

$$\|J^M_{\rho,A}(A(u)) - J^M_{\rho,A}(A(v))\| \leq \frac{s}{r - \rho m}\|u - v\|, \tag{2}$$

and

$$\langle J^M_{\rho,A}(A(u)) - J^M_{\rho,A}(A(v)), A(u) - A(v)\rangle \geq (r - \rho m)\|J^M_{\rho,A}(A(u)) - J^M_{\rho,A}(A(v))\|^2, \tag{3}$$

where $r - \rho m > 0$.

Proposition 2.8. *Let X be a real Hilbert space, let $H : X \to X$ be (r)-strongly monotone (s)-Lipschitz continuous, and let $M : X \to 2^X$ be H-maximal monotone. Then the generalized resolvent operator associated with M satisfies*

$$\|J^M_{\rho,H}(H(u)) - J^M_{\rho,H}(H(v))\| \leq \frac{s}{r}\|u - v\|, \tag{4}$$

and

$$\langle J^M_{\rho,H}(H(u)) - J^M_{\rho,H}(H(v)), H(u) - H(v)\rangle \geq r\|J^M_{\rho,H}(H(u)) - J^M_{\rho,H}(H(v))\|^2. \tag{5}$$

3. The new relaxed proximal point algorithm

This section deals with the introduction of a generalized version of the over-relaxed proximal point algorithm and its applications to approximation solvability of the inclusion problem (1) based on the A-maximal (m)-relaxed monotonicity.

Theorem 3.1. *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be A-maximal (m)-relaxed monotone. Then the following statements are equivalent:*

(i) *An element $u \in X$ is a solution to (1).*
(ii) *For an $u \in X$, we have*

$$u = J^M_{\rho,A}(A(u)),$$

where

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Theorem 3.2. *Let X be a real Hilbert space, let $H : X \to X$ be (r)-strongly monotone, and let $M : X \to 2^X$ be H-maximal monotone. Then the following statements are equivalent:*

(i) *An element $u \in X$ is a solution to (1).*
(ii) *For an $u \in X$, we have*

$$u = J^M_{\rho,H}(H(u)),$$

where

$$J^M_{\rho,H}(u) = (H + \rho M)^{-1}(u).$$

Theorem 3.3. *Let X be a real Hilbert space, let $A : X \to X$ be (r)-strongly monotone and (s)-Lipschitz continuous, and let $M : X \to 2^X$ be A-maximal (m)-relaxed monotone. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the generalized proximal point algorithm*

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k J^M_{\rho_k,A}(A(x^k)), \tag{6}$$

where $J^M_{\rho_k,A} = (A+\rho_k M)^{-1}$, and

$$\{\alpha_k\}, \{\rho_k\} \subseteq (0, \infty)$$

are scalar sequences. Suppose that there exists at least one solution to (1). If, in addition, for $\gamma > 1$,

$$\langle x^k - x^*, J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\rangle \geq \gamma \|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^2, \tag{7}$$

then the sequence $\{x^k\}$ converges linearly to a unique solution x^* of (1) with convergence rate

$$\theta_k = \sqrt{\left((1-\alpha_k)^2 + [\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma]\frac{s^2}{(r-\rho_k m)^2}\right)} < 1,$$

and $\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma > 0$ for $\alpha_k \geq 1$, $\gamma > 1$, $r - \rho m > 0$, $\alpha = \limsup_{k\to\infty} \alpha_k$, and $\rho = \limsup_{k\to\infty} \rho_k$.

Proof. Suppose that x^* is a zero of M for $x^* \in X$. From Theorem 3.1, it follows that x^* is a fixed point of $J^M_{\rho_k,A} oA$. Next, on applying (5), we find the estimate ($\alpha_k \geq 1$)

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|(1-\alpha_k)x^k + \alpha_k J^M_{\rho_k,A}(A(x^k)) - [(1-\alpha_k)x^* + \alpha_k J^M_{\rho_k,A}(A(x^*))]\|^2 \\
&= \|(1-\alpha_k)(x^k - x^*) + \alpha_k(J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*)))\|^2 \\
&= (1-\alpha_k)^2\|x^k - x^*\|^2 + 2\alpha_k(1-\alpha_k)\langle x^k - x^*, J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\rangle \\
&\quad + \alpha_k^2\|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^2 \\
&\leq (1-\alpha_k)^2\|x^k - x^*\|^2 + 2\alpha_k(1-\alpha_k)\gamma\|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^2 \\
&\quad + \alpha_k^2\|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^2 \\
&= (1-\alpha_k)^2\|x^k - x^*\|^2 + [\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma]\|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^2 \\
&\leq (1-\alpha_k)^2\|x^k - x^*\|^2 + [\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma]\frac{s^2}{(r-\rho_k m)^2}\|x^k - x^*\|^2 \\
&= \left((1-\alpha_k)^2 + [\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma]\frac{s^2}{(r-\rho_k m)^2}\right)\|x^k - x^*\|^2,
\end{aligned}$$

where $\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma > 0$ for $\alpha_k \geq 1$, $\gamma > 1$ and $r - \rho_k m > 0$.

Thus, we have

$$\|x^{k+1} - x^*\| \leq \theta_k\|x^k - x^*\|, \tag{8}$$

where

$$\theta_k = \sqrt{\left((1-\alpha_k)^2 + [\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma]\frac{s^2}{(r-\rho_k m)^2}\right)} < 1,$$

and $\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma > 0$ for $\alpha_k \geq 1$, $\gamma > 1$ and $r - \rho_k m > 0$. □

Corollary 3.1. *Let X be a real Hilbert space, let $H : X \to X$ be (r)-strongly monotone and (s)-Lipschitz continuous, and let $M : X \to 2^X$ be H-maximal monotone. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the generalized proximal point algorithm*

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k J^M_{\rho_k,H}(H(x^k)), \tag{9}$$

where $J^M_{\rho_k,H} = (H+\rho_k M)^{-1}$, and

$$\{\alpha_k\}, \{\rho_k\} \subseteq (0, \infty)$$

are scalar sequences. Suppose that $\{x^k\}$ is bounded in the sense that there exists at least one solution to (1). If, in addition, for $\gamma > 1$,

$$\langle x^k - x^*, J^M_{\rho_k,H}(H(x^k)) - J^M_{\rho_k,H}(H(x^*))\rangle \geq \gamma \|J^M_{\rho_k,H}(H(x^k)) - J^M_{\rho_k,H}(H(x^*))\|^2, \tag{10}$$

then the sequence $\{x^k\}$ *converges linearly to a unique solution* x^* *of* (1) *with the convergence rate*

$$\theta_k = \sqrt{\left((1-\alpha_k)^2 + [\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma]\frac{s^2}{r^2}\right)} < 1,$$

and $\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma > 0$ *for* $\alpha_k \geq 1$, *and* $\gamma > 1$, $\alpha = \limsup_{k\to\infty} \alpha_k$, *and* $\rho = \limsup_{k\to\infty} \rho_k$.

Remark 3.1. Note that the convergence rate estimate is strictly positive when $\alpha > 1, \gamma > 1$ and even if ρ becomes positive infinity, while for $\alpha = 1$ and $\rho_k \uparrow \infty$, the convergence rate estimate becomes superlinear on the basis of the assumptions of Theorem 3.3. It seems that the further convergence rates can be explored when $\alpha_k < 1$.

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